

A LOWER BOUND FOR THE CANONICAL HEIGHT ASSOCIATED TO A DRINFELD MODULE

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ABSTRACT. Denis associated to each Drinfeld module ϕ over a global function field L a canonical height function \hat{h}_ϕ , which plays a role analogous to that of the Néron-Tate height in the context of elliptic curves. We prove that there exists a constant $\varepsilon > 0$, depending only on the number of places at which ϕ has bad reduction, such that either $x \in \phi(L)$ is a torsion point of bounded order, or else

$$\hat{h}_\phi(x) \geq \varepsilon \max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\},$$

where j_ϕ and $\mathcal{D}_{\phi/L}$ are analogues of the j -invariant and minimal discriminant of an elliptic curve. As an application, we make some observations about specializations of one-parameter families of Drinfeld modules.

1. INTRODUCTION

Let X/\mathbb{F}_q be a curve, and let $A \subseteq K = \mathbb{F}_q(X)$ be the ring of functions which are regular at some specified point $\infty \in X(\mathbb{F}_q)$. If L/K is a finite extension, then a *Drinfeld A -module* ϕ/L is a ring homomorphism $\phi : A \rightarrow \text{End}_L(\mathbb{G}_a)$ satisfying certain additional conditions, and we denote the associated A -module structure on $\mathbb{G}_a(L)$ by $\phi(L)$. Denis [5] associated to such an object a canonical height function $\hat{h}_\phi : \phi(L) \rightarrow \mathbb{R}$, differing from the usual Weil height by at most a constant, and satisfying $\hat{h}_\phi(\phi_a(x)) = |a|_\infty^r \hat{h}_\phi(x)$ for each $x \in \phi(L)$ and $a \in A$, where $|\cdot|_\infty$ is the ∞ -adic absolute value, and $r \geq 1$ is the *rank* of ϕ . As a consequence of these two properties, the height \hat{h}_ϕ vanishes precisely on the torsion submodule of $\phi(L)$.

Our main result gives a lower bound on the canonical height associated to ϕ in terms of two quantities defined below, namely the j -invariant j_ϕ of ϕ , and the minimal discriminant $\mathcal{D}_{\phi/L}$, as well as the number of places at which ϕ has bad reduction. The quantities j_ϕ and $\mathcal{D}_{\phi/L}$ are defined by analogy to the case of elliptic curves, and Theorem 1.1 can be seen as an analogue of a result of Silverman [19] in that context (see also [10] in the context of the dynamics of unicritical polynomial maps). Indeed, just as in the results of [19], we need only count places at which ϕ has bad reduction that is not potentially good.

Theorem 1.1. *For every $s, r \geq 1$ there exist a constant $\varepsilon > 0$, and an ideal $\mathfrak{a} \subseteq A$, such that for every Drinfeld module ϕ/L of rank r with at most s places of persistently bad reduction, and every $x \in \phi(L)$, either $x \in \phi[\mathfrak{a}]$ or else*

$$\hat{h}_\phi(x) \geq \varepsilon \max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\}.$$

We immediately recover a theorem of Ghioca [7] bounding the torsion submodule of a Drinfeld module in terms of the number of places of bad reduction.

Corollary 1.2 (Ghioca [7]). *Let ϕ be a Drinfeld module over the global function field L . Then $\#\phi^{\text{Tors}}(L)$ is bounded by a quantity depending only on L , q , the rank of ϕ , and the number of places at which ϕ has bad reduction.*

We note that the result in [7] also involves a lower bound on the canonical height, and the methods introduced there contribute to the proof of Theorem 1.1. The lower bound produced in [7] is, however, of the form $\hat{h}_\phi(x) \geq \varepsilon$, where ε depends on the number of places of bad reduction, and hence is fundamentally weaker than that presented in Theorem 1.1. For example, we will show below that for any $B \geq 1$, there are only finitely many L -isomorphism classes of Drinfeld modules ϕ/L of a given rank satisfying

$$\max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\} \leq B.$$

The lower bound in Theorem 1.1, therefore, becomes arbitrarily large as one varies the Drinfeld module ϕ/L . The analogue of Theorem 1.1 for elliptic curves is a key ingredient in Silverman's quantitative version of Siegel's Theorem [23], and we suspect that Theorem 1.1 will have similar applications. Motivated by this, we propose the following conjecture, analogous to conjectures of Lang [22, Conjecture VIII.9.9] and Silverman [25, Conjecture 4.98].

Conjecture 1.3. *For every finite extension L/K and every $r \geq 1$ there is an $\varepsilon > 0$ such that if ϕ/L is a Drinfeld module of rank r , and $x \in \phi(L)$ is non-torsion, then*

$$\hat{h}_\phi(x) \geq \varepsilon \max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\}.$$

Another way to state of Theorem 1.1, then, is that Conjecture 1.3 holds if one restricts attention to Drinfeld modules with potentially good reduction at all but a bounded number of places. We note that, if true, Conjecture 1.3 is essentially the best-possible form of lower bound, at least if one assumes the weaker of Poonen's Uniform Boundedness Conjectures for Drinfeld modules [14, Conjecture 1].

Theorem 1.4. *Let L/K be a finite extension, let $r \geq 1$, and suppose that there is a uniform bound on the size of the torsion submodules $\phi^{\text{Tors}}(L)$ as ϕ/L varies over Drinfeld modules of rank r . Then for every Drinfeld module ϕ/L of rank r , there is a non-torsion $x \in \phi(L)$ such that*

$$\hat{h}_\phi(x) \leq 2 \max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\} + O(1),$$

where the implied constant depends only on L and r .

Note that, by Corollary 1.2, Theorem 1.4 can be made unconditional if one restricts attention to Drinfeld modules with potentially good reduction at all but a bounded number of places, showing that Theorem 1.1 is essentially the best possible result in this context.

As an application of Theorem 1.1, we consider the problem of specializing a family of Drinfeld modules. If C/L is a curve, and if $\phi/L(C)$ is a Drinfeld module, one might ask how the algebraic structure of the fibre $\phi_\beta(L)$ varies with $\beta \in C(L)$, omitting values at which the coefficients of ϕ are not defined. For example, if $x \in \phi(L(C))$ is non-torsion, then for each $\beta \in L(C)$ we obtain a specialization homomorphism $\sigma_\beta : \langle x \rangle \rightarrow \Gamma_{\phi_\beta}(x_\beta, L)$, where

$$\Gamma_{\phi_\beta}(x_\beta, L) = \{z \in \phi_\beta(L) : \phi_{\beta,a}(z) = \phi_{\beta,b}(x_\beta) \text{ for some } a, b \in A\}$$

is the largest submodule of $\phi_\beta(L)$ onto which one could hope that σ_β would surject, that is, the largest submodule of $\phi(L)$ containing x_β but having the same rank as $\langle x_\beta \rangle$. One could ask how often σ_β is injective, and how close σ_β is to being surjective.

Theorem 1.5. *Let $\phi/L(C)$ be a non-isotrivial Drinfeld A -module, and let $x \in \phi(L(C))$ be non-torsion.*

- (i) *The specialization homomorphism is injective outside of a set of bounded height.*
- (ii) *There exists an $E \in \text{Div}(C)$ and a finite set $S \subseteq M_L$ of places such that the index $(\Gamma_{\phi_\beta}(x_\beta, L) : \text{im}(\sigma_\beta))$ is bounded by a quantity which depends only on ϕ , x , and the number of places $v \notin S$ at which β is not integral relative to E .*

Theorem 1.5 is similar in flavour to two results of Silverman [20, 21] in the context of abelian varieties.

In the course of our study of local heights, we prove a result which may be of independent interest. Denis [5], in the paper defining the canonical height \hat{h}_ϕ associated to ϕ , shows that the difference between the canonical height and the usual height is bounded, and Theorem 1.6 makes this bound explicit. This is analogous to result for elliptic curves that was made explicit independently by Dem’janenko [4] and Zimmer [28], and may be useful for computations involving Drinfeld modules.

Theorem 1.6. *Let L be a global function field and let ϕ/L be a Drinfeld module of rank $r \geq 1$. There exist explicit constants C_1 and C_2 , depending only on A , L , and r , such that Then for all $x \in L^{\text{sep}}$, we have*

$$-C_1 - \left(\frac{1}{1 - q^{-r}} \right) h(\phi) \leq \hat{h}_\phi(x) - h(x) \leq h(\phi) + C_2,$$

where $h(\phi)$ is defined in Section 5. If L is a finite extension of the field K of fractions of A , and the heights above are relative to K , then C_1 and C_2 depend only on A and r .

With an eye to Theorem 1.1 and Conjecture 1.3, we note that while $h(\phi)$ is not the same as $\max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\}$, every Drinfeld module is L -isomorphic to a model for which these two quantities are commensurate.

In Section 2 we review some of the basic definitions of Drinfeld modules and heights, and introduce the j -invariant of a Drinfeld module. In Section 3 we look more closely at Drinfeld modules over local fields. Section 4 is devoted to the study of a local height function on a Drinfeld module over a local field, which is related to, but not the same as, the local heights introduced by Poonen [13]. Our local height functions have the advantage of being coordinate invariant, and more closely resembling the local heights on elliptic curves. In Section 5 we return to Drinfeld modules over global fields, defining the minimal discriminant $\mathcal{D}_{\phi/L}$, and proving Theorem 1.1, Theorem 1.4, and Theorem 1.6. Finally, in Section 6, we prove Theorem 1.5.

2. NOTATION AND PRELIMINARIES

2.1. Drinfeld modules. Throughout, we suppose that q is a power of a prime, and that K is a function field in one variable over \mathbb{F}_q , that is, that $K = \mathbb{F}_q(X)$ for some algebraic curve X/\mathbb{F}_q . We fix a place $\infty \in X(\mathbb{F}_q)$, and let $A \subseteq K$ denote the ring of regular functions at ∞ . By an A -field, we mean a field L with a homomorphism $i : A \rightarrow L$, and we will consider only the case in which L has *generic characteristic*, that is, where i is an injection; the typical example is where L/K is a finite extension, and i is the inclusion map.

If L is an A -field, then a *Drinfeld A -module* over L is a homomorphism $\phi : A \rightarrow \text{End}_L(\mathbb{G}_a)$, $a \mapsto \phi_a$, with the property that, for all $a \in A$,

$$\phi_a(x) = ax + O(x^q) \in L[x],$$

but $\phi_a(x) \neq ax$ for at least some $a \in A$. For $a \in A$, we let $|a|_\infty = \#(A/a)$, and recall that Drinfeld [6] proved the existence of an integer $r \geq 1$ such that $\deg(\phi_a(x)) = |a|_\infty^r$, for all $a \in A$. This quantity will be known as the *rank* of ϕ . For convenience we will define $\deg : A \rightarrow \mathbb{Z}$ by $|a|_\infty = q^{\deg(a)}$, noting that this agrees with the usual definition in the case $A = \mathbb{F}_q[T]$. If $\mathfrak{a} \subseteq A$ is any ideal, then we will write $\phi[\mathfrak{a}]$ for the submodule of $\phi(\overline{L})$ consisting of elements annihilated by \mathfrak{a} .

Two Drinfeld modules ϕ/L and ψ/L are said to be *isomorphic* over the extension E/L if and only if there exists an $\alpha \in E$ such that $\phi_a(\alpha x) = \alpha \psi_a(x)$ for all $a \in A$, abbreviated $\phi\alpha = \alpha\psi$. Suppose that we have fixed a ordered set of generators T_1, \dots, T_m for A as an \mathbb{F}_q -algebra. Then we have, for each i ,

$$\phi_{T_i}(x) = T_i + a_{i,1}x^q + \dots + a_{i,\deg(T_i)r}x^{q^{\deg(T_i)r}},$$

with $a_{i,j} \in L$, and ϕ is entirely determined by these values. By the \vec{w} -weighted projective space $\mathbb{P}^{\vec{w}}$, where $\vec{w} = (w_1, \dots, w_{m+1}) \in (\mathbb{Z}^+)^{m+1}$, we mean the quotient of $\mathbb{A}^{m+1} \setminus \{(0, 0, \dots, 0)\}$ under the $\mathbb{G}_m(\overline{L})$ action

$$(x_1, x_2, \dots, x_{m+1}) \rightarrow (\alpha^{w_1}x_1, \alpha^{w_2}x_2, \dots, \alpha^{w_{m+1}}x_{m+1}).$$

We warn the reader that, in general, points of $\mathbb{P}^{\vec{w}}$ which are fixed by $\text{Gal}(\overline{L}/L)$ do not necessarily have a representative with coordinates in L , unlike in the case of the usual projective space. The following definition is also made in [12].

Definition 2.1. Fix a set of generators T_1, \dots, T_m for A as an \mathbb{F}_q -algebra, fix $r \geq 1$, and let $M_{A,r}$ denote the weighted projective space with coordinates $x_{i,j}$, for $1 \leq i \leq m$ and $1 \leq j \leq r \deg(T_i)$, such that $x_{i,j}$ is given weight $q^j - 1$. If ϕ/L is a Drinfeld A -module of rank $r \geq 1$, then by the j -invariant of ϕ/L , we mean the point

$$j_\phi = [a_{1,1}, a_{1,2}, \dots, a_{1,\deg(T_1)r}, a_{2,1}, \dots, a_{2,\deg(T_2)r}, \dots, a_{m,1}, a_{m,2}, \dots, a_{m,\deg(T_m)r}]$$

in $M_{A,r}(L)$.

If $A = \mathbb{F}_q[T]$, and ϕ/L is the Drinfeld module of rank 2 defined by

$$\phi_T(x) = Tx + gx^q + \Delta x^{q^2},$$

then it is conventional to set $j = g^{q+1}/\Delta$. We note that this is the image of the point j_ϕ , as defined here, under the map $\mathbb{P}^{(q-1, q^2-1)} \rightarrow \mathbb{P}^1$ given by $[x : y] \mapsto [x^{q+1} : y]$. So our j -invariant nearly exactly generalizes the conventional notion in the rank 2 case. Note that our invariant is similar, but not identical, to the j -invariant defined by Potemine [15].

Lemma 2.2. *Let ϕ/L and ψ/L be two Drinfeld A -modules. Then the following are equivalent:*

- (i) ϕ and ψ are L^{sep} -isomorphic;
- (ii) ϕ and ψ are \overline{L} -isomorphic;
- (iii) $j_\phi = j_\psi$.

Proof. Since $L^{\text{sep}} \subseteq \overline{L}$, it is clear that condition (1) implies condition (2). Suppose that (2) is true, and that $\alpha \in \overline{L}$ satisfies $\alpha\psi = \phi\alpha$. Then if

$$\phi_{T_i}(x) = T_i x + a_{i,1}x^q + \cdots + a_{i,\deg(T_i)r}x^{q^{\deg(T_i)r}}$$

and

$$\psi_{T_i}(x) = T_i x + b_{i,1}x^q + \cdots + b_{i,\deg(T_i)r}x^{q^{\deg(T_i)r}},$$

one checks easily that $b_{i,j} = \alpha^{q^j-1}a_{i,j}$ for all i and j . This is precisely what it means for two points in $\mathbb{P}^{\vec{w}}$ to be equal.

Finally, suppose that $j_\phi = j_\psi$. Then if the $a_{i,j}$ and $b_{i,j}$ are defined as above, there exists an $\alpha \in \overline{L}^*$ such that $a_{i,j} = \alpha^{q^j-1}b_{i,j}$, for all i and j . Choosing some i and j with $b_{j,i} \neq 0$, we see that α is a root of the polynomial $b_{i,j}X^{q^j-1} - a_{i,j} \in L[X]$, and so in particular $\alpha \in L^{\text{sep}}$. One now confirms easily that $\alpha\psi_a(x) = \phi_a(\alpha x)$ for all $a \in \{T_1, \dots, T_m\}$, and hence the same is true for all $a \in A$, since T_1, \dots, T_m generate A over \mathbb{F}_q . It follows that $\alpha\psi = \phi\alpha$, and hence that ϕ and ψ are L^{sep} -isomorphic. \square

2.2. Heights and valuations. Throughout, we will make the convention of normalizing logarithms so that $\log q = 1$, and write $\log^+ x$ for $\max\{0, \log x\}$. Let L be a *global function field*, by which we mean a field equipped with a set of non-trivial, non-archimedean absolute values M_L satisfying the product formula:

$$\sum_{v \in M_L} \log |x|_v = 0$$

for all $x \in L^*$. If L'/L is a finite extension, we normalize the elements of $M_{L'}$ so that if $w \in M_L$ is the place below $v \in M_{L'}$, then $|x|_v = |x|_w$ for all $x \in L$. A finite extension L'/L will be called *reasonable* if

$$\sum_{\substack{w \in M_{L'} \\ w|v}} [L'_v : L_v] = [L' : L]$$

for every place $v \in M_{L'}$, and we note that every separable extension is reasonable (see [18, p. 13]). Since composita of reasonable extensions are again reasonable, we will define the *reasonable closure* L^c to be the union of all finite reasonable extensions of L . Note that $L^{\text{sep}} \subseteq L^c \subseteq \overline{L}$. The typical case will be that in which L/K is a finite extension, in which case we have $L^c = \overline{L}$.

Fixing a *ground field* L , if L'/L is a finite reasonable extension, then for any $x \in L'$, we will define the *height* of x to be

$$h(x) = \sum_{v \in M_{L'}} \frac{[L'_v : L_v]}{[L' : L]} \log^+ |x|_v,$$

where L_v , as usual, denotes the completion of L at v . Note that h extends to a well-defined function $h : L^c \rightarrow \mathbb{R}$.

More generally, if $\mathbb{P}^{\vec{w}}$ is the weighted projective space with weights $\vec{w} = (w_0, \dots, w_N)$ (all non-zero), then we define a height on $\mathbb{P}^{\vec{w}}$ by

$$h([x_0 : \cdots : x_N]) = \sum_{v \in M_{L'}} \frac{[L'_v : L_v]}{[L' : L]} \log \max\{|x_0|_v^{1/w_0}, \dots, |x_N|_v^{1/w_N}\}.$$

Note that, considering the morphism $\Phi : \mathbb{P}^{\vec{w}} \rightarrow \mathbb{P}^N$ given by

$$\Phi([x_0 : x_1 : \cdots : x_N]) = [x_0^{w_1 w_2 \cdots w_N} : x_1^{w_0 w_2 \cdots w_N} : \cdots : x_N^{w_0 w_1 \cdots w_{N-1}}],$$

we see that sets of bounded height in $\mathbb{P}^w(L)$ map to sets of bounded height in $\mathbb{P}^N(L)$, and hence are finite. In particular, the standard Northcott finiteness property for \mathbb{P}^N yields the following useful observation.

Proposition 2.3. *For any $B \geq 1$, there are only finite many \bar{L} -isomorphism classes of ϕ/L with $h(j_\phi) \leq B$.*

3. DRINFELD MODULES OVER LOCAL FIELDS

3.1. Green's functions. Let L be a complete local field, equipped with the non-trivial non-archimedean absolute value $|\cdot|$. To avoid confusion, we will denote by $|\cdot|_\infty$ the absolute value on $A \subseteq K$ corresponding to the place at ∞ . For any polynomial

$$f(x) = a_0 + a_1x + \cdots + a_dx^d \in L[x],$$

where $d \geq 2$ and $a_d \neq 0$, we define the *Green's function associated to f* to be the real-valued function defined by

$$(1) \quad G_f(x) = \lim_{N \rightarrow \infty} d^{-N} \log^+ |f^N(x)|.$$

The following lemma is standard.

Lemma 3.1. *Let $f(x) \in L[x]$ be a polynomial as above, and let G_f be the Green's function associated to f . Then*

- (i) $G_f : L \rightarrow \mathbb{R}$ is everywhere defined and non-negative;
- (ii) $G_f(f(x)) = \deg(f)G_f(x)$ for all $x \in L$;
- (iii) $G_f(x) = \log|x| + \frac{1}{d-1} \log|a_d|$ whenever

$$(2) \quad |x| > \log \max \left\{ |a_d|^{-1/(d-1)}, |a_i/a_d|^{1/(d-i)} \right\}.$$

Proof. The non-negativity of G_f , where it is defined, follows simply from the fact that it is a pointwise limit of non-negative functions. It is easy to check statement (iii), since the condition (2) ensures that $|f(x)| = |a_dx^d| > |x|$. The fact that the limit (1) exists for all x is now straightforward, since if the condition (2) occurs for $f^N(x)$, for any N , we have an explicit value for the limit, and otherwise it is clear that $G_f(x) = 0$. Part (ii) follows directly from the definition, once we know that the limit always exists. \square

Now let L be a complete local A -field, and let ϕ/L be a Drinfeld A -module of rank r . Then for each non-constant $a \in A$, we have $\deg(\phi_a) = |a|_\infty^r \geq 2$, and so we may associate to the polynomial $\phi_a(x)$ a Green's function G_{ϕ_a} . The following lemma is well-known, although we include a short proof for the convenience of the reader. It essentially Proposition 3 of [13], in different notation, or the well-known fact in holomorphic dynamics that commuting maps have identical Green's functions.

Lemma 3.2. *For any non-constant $a, b \in A$, we have $G_{\phi_a} = G_{\phi_b}$.*

Proof. First note that there exists a constant C , depending on b , such that

$$\deg(\phi_b) \log^+ |x| - C \leq \log^+ |\phi_b(x)|_v \leq \deg(\phi_b) \log^+ |x| + C.$$

Since ϕ_a and ϕ_b commute, it follows that

$$\begin{aligned} G_{\phi_a}(\phi_b(x)) &= \lim_{N \rightarrow \infty} \deg(\phi_a)^{-N} \log^+ |\phi_a^N(\phi_b(x))| \\ &= \lim_{N \rightarrow \infty} \deg(\phi_a)^{-N} \log^+ |\phi_b(\phi_a^N(x))| \\ &= \lim_{N \rightarrow \infty} \deg(\phi_a)^{-N} \deg(\phi_b) \log^+ |\phi_a^N(x)| \\ &= \deg(\phi_b) G_{\phi_a}(x). \end{aligned}$$

But $G_{\phi_a}(x)$ and $\log^+ |x|$ also differ only by a bounded amount, by the previous lemma, and so we also have

$$\begin{aligned} G_{\phi_b}(x) &= \lim_{N \rightarrow \infty} \deg(\phi_b)^{-N} \log^+ |\phi_b^N(x)| \\ &= \lim_{N \rightarrow \infty} \deg(\phi_b)^{-N} G_{\phi_a}(\phi_b^N(x)) \\ &= \lim_{N \rightarrow \infty} \deg(\phi_b)^{-N} \deg(\phi_b)^N G_{\phi_a}(x) \\ &= G_{\phi_a}(x). \end{aligned}$$

□

We now know that the Green's function G_{ϕ_T} is independent of the choice of $T \in A \setminus \mathbb{F}_q$, and so we may define the *Green's function associated to ϕ* to be the function $G_\phi = G_{\phi_T}$, for any non-constant $T \in A$. One important corollary of this is that the quantity

$$\lim_{|x| \rightarrow \infty} (\log |x| - G_{\phi_T}(x)) = \frac{1}{q^{\deg(T)} - 1} \log |a_{r \deg(T)}^{-1}|,$$

which would initially appear to depend on the choice of $T \in A \setminus \mathbb{F}_q$, is in fact independent of this choice. We will denote this quantity by $c(\phi)$. Note that Theorem 4.1 of [2] shows that $c(\phi)$ is the logarithmic transfinite diameter of filled Julia set of ϕ/\mathbb{C}_v , where \mathbb{C}_v is the completion of the algebraic closure of L , which is also the logarithmic capacity of the Berkovich filled Julia set (see [12] for more details).

We note two properties, also present in [13], which are easy to prove from the definitions.

Lemma 3.3. *For all $x, y \in L$ and all $a \in A$, we have*

$$G_\phi(x + y) \leq \max\{G_\phi(x), G_\phi(y)\},$$

with equality unless $G_\phi(x) = G_\phi(y)$, and

$$G_\phi(\phi_a(x)) = |a|_\infty^r G_\phi(x).$$

Before proceeding to our discussion of local heights, we introduce two more pieces of notation, which also appear in [12]. Fix a basis T_1, \dots, T_m for A as an \mathbb{F}_q -algebra, as above. For a Drinfeld module ϕ/L , let $a_{i,j} \in L$ be defined by

$$\phi_{T_i}(x) = T_i x + a_{i,1} x^q + \dots + a_{i,r \deg(T_i)} x^{q^{t \deg(T_i)}},$$

for each i , and set

$$j_{\phi,v} = \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq r \deg(T_i)}} \left\{ \frac{1}{q^j - 1} \log |a_{i,j}| \right\} + c(\phi).$$

In the context of a global function field, $j_{\phi,v}$ will be the local contribution to the height of j_ϕ . We note, as proven in [12], that $j_{\phi,v}$ is non-negative, and that for

finite places v we have $j_{\phi,v} = 0$ if and only if ϕ has potentially good reduction at v . Similarly, for each non-constant $T \in A$, we define $j_{\phi_T,v}$ by

$$j_{\phi_T,v} = \max_{1 \leq j \leq r \deg(T)} \left\{ \frac{1}{q^j - 1} \log |a_j| \right\} + c(\phi).$$

Finally, for any ϕ/L we define $B_T > 0$ by

$$\log B_T = \log \max\{|\xi| : \xi \in \phi[T]\} + \frac{1}{q^{r \deg(T)} - 1} \log^+ |T^{-1}|.$$

We note that, by Lemma 3.1, if $|x| > B_T$, we have $G_\phi(x) = \log |x| - c(\phi)$.

We end this section with a convenient lemma from [12]. In this lemma, and the next section, we will say that the valuation v on L is a *finite place* if the images of the elements of A in L are all v -integral, and we will call v an *infinite place* otherwise.

Lemma 3.4. *If v is a finite place, then $j_{\phi,v} = j_{\phi_T,v}$, for any non-constant $T \in A$.*

Proof. The proof of this fact, which follows from an argument similar to that giving the Gauss Lemma, is found in [12]. \square

3.2. The (local) minimal discriminant. In this section we define a quantity $\mathcal{D}_{\phi,v}$, which will become the local contribution to the minimal discriminant $\mathcal{D}_{\phi/L}$ in Section 5. We let L be a complete, local A -field with valuation v , and assume additionally that the ring $\mathcal{O} \subseteq L$ of integral elements is a normalized discrete valuation ring. As usual, we will let $\mathbb{C}_v \supseteq L$ be the completion of the algebraic closure of L .

Definition 3.5. We say that a Drinfeld module ψ/L is an *integral model* of ϕ/L if ψ is L -isomorphic to ϕ , and if $\psi_a(x) \in \mathcal{O}[x]$ for every $a \in A$. We define $\mathcal{D}_{\phi,v}$ by

$$\mathcal{D}_{\phi,v} = \min\{c(\psi) : \psi/L \text{ is an integral model of } \phi\}.$$

We will call ψ/L a *minimal model* of ϕ/L if it is an integral model, and $\mathcal{D}_{\phi/L} = c(\psi)$.

Note that, if π is a uniformizer for v , then $\pi^{-N} \phi \pi^N$ is an integral model for N sufficiently large, and so the set of which we are taking the minimum is not vacuous. On the other hand, v is discrete and $c(\psi) \geq 0$ for every integral model of ϕ , so the minimum in the definition exists.

Proposition 3.6. *If v is a finite place, then*

$$j_{\phi,v} \leq \mathcal{D}_{\phi,v} < j_{\phi,v} + \deg(v).$$

In particular, if $j_{\phi,v} > 0$ then

$$j_{\phi,v} > \frac{1}{d+1} \max\{j_{\phi,v} + \mathcal{D}_{\phi,v}\}$$

for some $d \geq 1$ depending only on A and the rank of ϕ .

Proof. Note that $j_{\phi,v}$ is a \mathbb{C}_v -isomorphism invariant, and so it suffices to show that

$$j_{\psi,v} \leq c(\psi) < j_{\psi,v} + \deg(v),$$

where ψ is an integral model for ϕ/L . The first inequality is trivial from the definition of $j_{\psi,v}$, and the fact that the coefficients of $\psi_a(x)$ are integral for every $a \in A$. Now suppose, toward a contradiction, that $c(\psi) \geq j_{\psi,v} + \deg(v)$ and that

π is a uniformizer for v . Then, if T_1, \dots, T_m is the generated set for A used in the definition of $j_{\psi, v}$, we have $\frac{1}{q^j - 1} \log |a_{i,j}| \leq -\deg(v)$, for all i and j , where

$$\psi_{T_i}(x) = T_i x + a_{i,1} x^q + \dots + a_{i, \deg(T_i)r} x^{q^{\deg(T_i)r}}.$$

In particular, if

$$\pi \psi \pi_{T_i}^{-1}(x) = T_i x + b_{i,1} x^q + \dots + b_{i, \deg(T_i)r} x^{q^{\deg(T_i)r}}$$

for all i , then $\frac{1}{q^j - 1} \log |b_{i,j}| \leq 0$. In other words, $\pi \psi \pi^{-1}$ is an integral model of ϕ . But $c(\pi \psi \pi^{-1}) < c(\psi)$, contradicting the minimality of ψ . It must be the case, then, that $\mathcal{D}_{\phi, v} < j_{\phi, v} + \deg(v)$.

Now, suppose that $j_{\phi, v} > 0$. Note that if $M = \max\{\deg(T_1), \dots, \deg(T_m)\}$, where the T_i are our fixed generators of A as an \mathbb{F}_q -algebra, and

$$d = d(A, r) = \text{lcm}\{q^j - 1 : 1 \leq j \leq rM\},$$

then $j_{\phi, v} \in \deg(v)\mathbb{Z}[1/d]$. In particular, $j_{\phi, v} = 0$ or $j_{\phi, v} \geq \deg(v)/d$. It follows, if $j_{\phi, v} \neq 0$, that

$$\mathcal{D}_{\phi, v} < j_{\phi, v} + \deg(v) \leq (1 + d)j_{\phi, v},$$

whence the bound. \square

4. LOCAL HEIGHTS FOR DRINFELD MODULES

In this section we lay out a theory of local heights for Drinfeld modules. Local heights for Drinfeld modules have previously been considered by Poonen [13] and Ghioca [7, 8], but we define them slightly differently here. Throughout this section, \mathbb{C}_v will be a complete, algebraically closed A -field with valuation v .

Definition 4.1. Let ϕ/\mathbb{C}_v be a Drinfeld module. Define the *local height* associated to ϕ by

$$\lambda_{\phi}(x) = \log |x^{-1}| + G_{\phi}(x) + c(\phi).$$

Remark 4.2. We note that λ admits a natural extension to the Berkovich analytic space $\mathbb{A}_{\text{Berk}}^1$ associated to \mathbb{C}_v (see [3]), namely

$$\lambda_{\phi}(x) = -\log \delta(x, 0)_{\infty} + G_{\phi}(x) + c(\phi),$$

where $\delta(x, y)_{\infty}$ is the Hsia kernel [3, p. 73] and where G_{ϕ} is extended in the natural way. In particular, one can check that this extension of $\lambda_{\phi}(x)$ agrees with $g_{\mu_{\phi}}(x, 0)$, where $g_{\mu_{\phi}}(x, y)$ is the Arakelov-Green's function on $\mathbb{P}_{\text{Berk}}^1 \times \mathbb{P}_{\text{Berk}}^1$ for the invariant measure associated to $\phi_T(x)$, for any non-constant $T \in A$ [3, p. 300]. In some sense, then, $\lambda_{\phi}(x)$ is the correct local height from the capacity-theoretic viewpoint, relative to the identity element of the module. Some of the properties of λ_{ϕ} outlined in Lemma 4.3 below may be deduced from the properties of $g_{\mu_{\phi}}$ derived in [3], but we give direct and elementary proofs for completeness and simplicity.

Lemma 4.3. Let λ_{ϕ} be the local height defined above, for the Drinfeld module ϕ/\mathbb{C}_v .

- (i) The map $\lambda_{\phi} : \phi(\mathbb{C}_v) \rightarrow \mathbb{R}$ is continuous, except for a simple pole at the origin.
- (ii) If $\psi\alpha = \alpha\phi$, for some $\alpha \in \mathbb{C}_v^*$, then

$$\lambda_{\phi}(x) = \lambda_{\psi}(\alpha x).$$

(iii) If ϕ has good reduction, then

$$\lambda_\phi(x) = \log^+ |x^{-1}|.$$

(iv) If ϕ has potentially good reduction, then λ_ϕ is non-negative.

(v) If $|x| > B_T$, for any $T \in A \setminus \mathbb{F}_q$ then

$$\lambda_\phi(x) = 0.$$

(vi) For any $a \in A$, we have

$$\lambda_\phi(\phi_a(x)) = |a|_\infty^r \lambda_\phi(x) - \log \left| \frac{\phi_a(x)}{\Delta x^{|a|_\infty^r}} \right|,$$

where Δ is the leading coefficient of $\phi_a(x)$.

Remark 4.4. Compare with Theorem 4.2 of [24, p. 473] for local heights relative to elliptic curves over p -adic fields.

Proof of Lemma 4.3. (i) This result is clear enough, but also follows from [3, Proposition 8.66 p. 242].

(ii) If $T \in A$ is non-constant, and

$$\psi_T(x) = \alpha \phi_T(\alpha^{-1}z),$$

we have

$$G_{\phi_T}(z) = G_{\psi_T}(\alpha z)$$

and $c(\psi) = c(\phi) - \log |\alpha|_v$, so

$$\begin{aligned} \lambda_\phi(x) &= \log |x^{-1}| + G_{\phi_T}(x) + c(\phi) \\ &= \log |x^{-1}| + G_{\psi_T}(\alpha x) + c(\psi) - \log |\alpha| \\ &= \lambda_\psi(\alpha x). \end{aligned}$$

So the local height is coordinate invariant.

(iii) Note that if ϕ/\mathbb{C}_v has good reduction, then the coefficients of ϕ_T are integral, and the leading coefficient is a unit. It follows that $c(\phi) = 0$, and from Lemma 3.1 that $G_{\phi_T}(x) = \log^+ |x|$. So

$$\lambda_\phi(x) = \log |x^{-1}| + \log^+ |x| = \log^+ |x^{-1}|.$$

(iv) If ϕ has potentially good reduction then there is a ψ/\mathbb{C}_v with good reduction, and an $\alpha \in \mathbb{C}_v^*$ such that $\psi\alpha = \alpha\phi$. By two of the claims above, we have

$$\lambda_\phi(x) = \log^+ |(\alpha x)^{-1}| \geq 0$$

for all x .

(v) This follows directly from Lemma 3.1.

(vi) This is proven by noting that

$$\begin{aligned} \lambda_\phi(\phi_a(x)) &= \log |\phi_a(x)^{-1}| + G_{\phi_a}(\phi_a(x)) - \frac{1}{|a|_\infty^r - 1} \log |\Delta| \\ &= |a|_\infty^r \log |x^{-1}| + |a|_\infty^r G_{\phi_a}(x) - \frac{|a|_\infty^r}{|a|_\infty^r - 1} \log |\Delta| - \log |\phi_a(x)| \\ &\quad + |a|_\infty^r \log |x| + \frac{|a|_\infty^r - 1}{|a|_\infty^r - 1} \log |\Delta| \\ &= |a|_\infty^r \lambda_\phi(x) - \log \left| \frac{\phi_a(x)}{\Delta x^{|a|_\infty^r}} \right|. \end{aligned}$$

□

We now proceed with some definitions and lemmas aimed at showing that one cannot have too large a set of points in $\phi(L)$ at which both λ_ϕ and G_ϕ take small values. The following definition and lemmas are motivated by the work of Ghioca [7, 8].

Definition 4.5. If ϕ/L is a Drinfeld module, and $T \in A$ is non-constant, then we say that $x \in L$ is *T-generic* if

$$|\phi_T(x)| = \max_{0 \leq i \leq r \deg(T)} |a_i x^{q^i}|,$$

where $a_0 = T$ as usual.

Lemma 4.6. *If $x \in \phi(\mathbb{C}_v)$ is T-generic and*

$$\log |\phi_T(x)| \leq \log B_T$$

then

$$\log |x| \leq c(\phi) + \frac{1}{(q^{r \deg(T)} - 1)^2} \log^+ |T^{-1}|.$$

Remark 4.7. Note that $c(\phi)$ is the average (logarithmic) size of the non-zero elements of $\phi[T]$, and so the lemma says that *T-generic* points whose image under ϕ_T is no larger than the largest elements of $\phi[T]$ are themselves no larger than the average elements of $\phi[T]$.

Proof of Lemma 4.6. To begin, write

$$\phi_T(x) = Tx + a_1 x^q + \cdots + \Delta x^{q^{r \deg(T)}},$$

where we take $a_0 = T$ and $a_{r \deg T} = \Delta$. We will, for convenience, write $R = r \deg(T)$. Using the hypothesis that x is *T-generic*, we choose i so that

$$|\phi_T(x)| = |a_i x^{q^i}| = \max_{0 \leq k \leq R} \left\{ |a_k x^{q^k}| \right\}.$$

From the theory of Newton polygons, we see that we may also choose $0 \leq j < R$ such that

$$\frac{1}{q^R - q^j} \log |a_j / \Delta| + \frac{1}{q^R - 1} \log^+ |T^{-1}| = \log B_T.$$

We proceed much as in [7]. By hypothesis

$$q^i \log |x| + \log |a_i| \leq \log B_T = \frac{1}{q^R - q^j} \log |a_j / \Delta| + \frac{1}{q^R - 1} \log^+ |T^{-1}|.$$

Re-arranging this, we have

$$(q^{R+i} - q^{j+i}) \log |x| + (q^R - q^j) \log |a_i| \leq \log |a_j| - \log |\Delta| + \frac{q^R - q^j}{q^R - 1} \log^+ |T^{-1}|.$$

Also, by hypothesis, we have

$$|a_i x^{q^i}| \geq |a_j x^{q^j}|,$$

and so we may conclude that

$$\log |a_j| \leq \log |a_i| + (q^i - q^j) \log |x|.$$

Combining these gives

$$\begin{aligned} (q^{R+i} - q^{j+i}) \log |x| + (q^R - q^j) \log |a_i| \\ \leq \log |a_i| + (q^i - q^j) \log |x| - \log |\Delta| + \frac{q^R - q^j}{q^R - 1} \log^+ |T^{-1}| \end{aligned}$$

or

$$(3) \quad (q^{R+i} - q^{j+i} - q^i + q^j) \log |x| \leq (1 - q^R + q^j) \log |a_i| - \log |\Delta| + \frac{q^R - q^j}{q^R - 1} \log^+ |T^{-1}|.$$

At the same time, our hypotheses also ensure that

$$|a_i x^{q^i}| \geq |\Delta x^{q^R}|,$$

and so

$$\log |a_i| \geq (q^R - q^i) \log |x| + \log |\Delta|.$$

But $q \geq 2$, and hence $q^j + 1 \leq q^{j+1} \leq q^R$, so we have $1 - q^R + q^j \leq 0$. This gives

$$(1 - q^R + q^j) \log |a_i| \leq (1 - q^R + q^j) ((q^R - q^i) \log |x| + \log |\Delta|).$$

Combining this with (3) gives

$$\begin{aligned} (q^{R+i} - q^{j+i} - q^i + q^j) \log |x| \\ \leq (1 - q^R + q^j) ((q^R - q^i) \log |x| + \log |\Delta|) - \log |\Delta| + \frac{q^R - q^j}{q^R - 1} \log^+ |T^{-1}|, \end{aligned}$$

whereupon

$$(q^R - 1)(q^R - q^j) \log |x| \leq (q^R - q^j) \log |\Delta^{-1}| + \frac{q^R - q^j}{q^R - 1} \log^+ |T^{-1}|.$$

Recalling that $c(\phi) = \frac{1}{q^R - 1} \log |\Delta^{-1}|$, this ensures that

$$\log |x| \leq c(\phi) + \frac{1}{(q^R - 1)^2} \log^+ |T^{-1}|.$$

□

Lemma 4.8. *If $x \in \phi(\mathbb{C}_v)$ is T -generic and*

$$\log |\phi_T(x)| \leq c(\phi) + \frac{1}{(q^{r \deg(T)} - 1)^2} \log^+ |T^{-1}|,$$

then

$$-\log |x|_v + c(\phi) \geq (1 - q^{-1}) j_{\phi_T, v} - \frac{1}{q(q^{r \deg(T)} - 1)^2} \log^+ |T^{-1}|.$$

Proof. Suppose that x is generic, and that $\log |\phi_T(x)| \leq c(\phi)$. Then we have

$$\log |\phi_T(x)| = \max_{0 \leq i \leq r \deg(T)} \left\{ \log |a_i x^{q^i}| \right\} \geq \log |a_k x^{q^k}|,$$

where $1 \leq k \leq r \deg(T)$ is an index maximizing $\frac{1}{q^k - 1} \log |a_k|$. Note that, by definition,

$$j_{\phi_T, v} = \frac{1}{q^k - 1} \log |a_k| + c(\phi).$$

It follows that

$$\begin{aligned} c(\phi) + \frac{1}{(q^{r \deg(T)} - 1)^2} \log^+ |T^{-1}| &\geq \log |\phi_T(x)| \\ &\geq \log |a_k| + q^k \log |x| \\ &= (q^k - 1)(j_{\phi_T, v} - c(\phi)) + q^k \log |x|, \end{aligned}$$

and hence that

$$\begin{aligned} -\log |x|_v + c(\phi) &\geq -q^{-k} \left(c(\phi) + \frac{1}{(q^{r \deg(T)} - 1)^2} \log^+ |T^{-1}| - (q^k - 1)(j_{\phi_T, v} - c(\phi)) \right) + c(\phi) \\ &= (1 - q^{-k})j_{\phi_T, v} - q^{-k} \frac{1}{(q^{r \deg(T)} - 1)^2} \log^+ |T^{-1}| \\ &\geq (1 - q^{-1})j_{\phi_T, v} - \frac{1}{q(q^{r \deg(T)} - 1)^2} \log^+ |T^{-1}|, \end{aligned}$$

since $j_{\phi, v}$ and $\log^+ |T^{-1}|$ are non-negative. \square

The final lemma of this section shows, essentially, that we cannot have too large a collection of values $a \in A$ such that both $G_\phi(\phi_a(x))$ and $\lambda_\phi(\phi_a(x))$ are small. This will help us establish our lower bounds in Section 5, since the sum of $G_\phi(x)$ over all places agrees with the same sum for $\lambda_\phi(x)$, as long as x is non-zero.

Lemma 4.9. *Suppose that $X \subseteq A$ is an additive subgroup, and that for all $a \in X$ we have*

$$\log |\phi_{T^2 a}(x)| \leq \log B_T.$$

Then there is an additive subgroup $Y \subseteq X$ with $\#Y \geq q^{-4r^2 \deg(T)^2} \#X$ such that for all $a \in Y$, we have either $\phi_a(x) = 0$, or

$$-\log |\phi_a(x)| + c(\phi) \geq (1 - q^{-1})j_{\phi_T, v} - \frac{1}{q(q^{r \deg(T)} - 1)^2} \log^+ |T^{-1}|.$$

Proof. To begin, we note that it suffices to construct a subset $Y \subseteq X$ satisfying the criteria, since the inequality

$$-\log |x + y| \geq \min\{-\log |x|, -\log |y|\}$$

will ensure that the subgroup generated by Y will also satisfy the criteria. Second of all, we note that it suffices to find a subset $Y \subseteq X$ of the appropriate size, such that for all $a \in Y$, both $\phi_a(x)$ and $\phi_{Ta}(x)$ are T -generic, since $\log |\phi_{T^2 a}(x)| \leq \log B_T$ implies $\log |\phi_{Ta}(x)| \leq \log B_T$, and so we may apply Lemma 4.6 to $\phi_{Ta}(x)$ and then Lemma 4.8 to obtain the requisite bound on $|\phi_a(x)|$.

For any set $W \subseteq A$, we define $N_1(W)$ so that $q^{N_1(W)}$ is the number of elements $\xi \in \phi[T]$ such that $|\xi| \leq \max\{|\phi_a(x)| : a \in W\}$. Note that, by examining the Newton polygon of $\phi_T(x)$, we see that $N_1(W)$ is always an integer, and $0 \leq N_1(W) \leq R$ as long as W is non-empty, where as usual we take $R = r \deg(T)$. Similarly, define $N_T(W) = N_1(TW)$, that is, $q^{N_T(W)}$ is the number of $\xi \in \phi[T]$ such that $|\xi| \leq \max\{|\phi_{Ta}(x)| : a \in W\}$.

For $\xi \in \phi[T]$, define

$$Z_\xi = \{y \in \phi(\mathbb{C}_v) : |y - \xi| < |\xi| = |y|\}$$

if $\xi \neq 0$, and

$$Z_0 = \{y \in \phi(\mathbb{C}_v) : y \text{ is generic}\}.$$

Note that $\phi(\mathbb{C}_v) = \bigcup_{\xi \in \phi[T]} Z_\xi$. Let $X_1 = X$, and suppose that for $i \geq 1$ we have constructed a set X_i . For each $(\xi_1, \xi_2) \in \phi[T] \times \phi[T]$, let

$$X_{i,\xi_1,\xi_2} = \{a \in X_i : \phi_a(x) \in Z_{\xi_1} \text{ and } \phi_{Ta}(x) \in Z_{\xi_2}\}.$$

Clearly the sets X_{i,ξ_1,ξ_2} cover X_i , and there are $(q^R)^2$ of them, so there is some pair (ξ_1, ξ_2) such that $\#X_{i,\xi_1,\xi_2} \geq q^{-2R} \#X_i$.

We consider several cases.

Case 1: $\xi_1 \neq 0$.

Choose some $a \in X_{i,\xi_1,\xi_2}$, and let $X_{i+1} = X_{i,\xi_1,\xi_2} - a$, so that $\#X_{i+1} = \#X_{i,\xi_1,\xi_2}$. Since

$$\max\{|\phi_{Tb}(x) - \phi_{Ta}(x)| : b \in X_{i,\xi_1,\xi_2}\} \leq \max\{|\phi_{Tb}(x)| : b \in X_i\},$$

we see that $N_T(X_{i+1}) \leq N_T(X_i)$. On the other hand,

$$|\phi_b(x) - \phi_a(x)| = |\phi_b(x) - \xi_1 + \phi_a(x) - \xi_1| < |\xi_1| = |\phi_b(x)| = |\phi_a(x)|$$

for all $a, b \in X_{i,\xi_1,\xi_2}$, by definition. It follows that

$$N_1(X_{i+1}) < N_1(X_{i,\xi_1,\xi_2}) \leq N_1(X_i).$$

In particular, since $N_1(W)$ is a non-negative integer for any non-empty $W \subseteq A$, the present case can arise for at most R values of i .

Case 2: $\xi_1 = 0, \xi_2 \neq 0$.

In this case we again choose some $a \in X_{i,\xi_1,\xi_2}$, and let $X_{i+1} = X_{i,\xi_1,\xi_2} - a$. Arguments just as in Case 1 show that here $N_1(X_{i+1}) \leq N_1(X_i)$, while $N_T(X_{i+1}) < N_T(X_i)$. In particular, this case can also arise for at most R values of i .

Case 3: $\xi_1 = \xi_2 = 0$.

In this case, we may take $Y = X_{i,0,0}$, since then $\phi_a(x)$ and $\phi_{Ta}(x)$ are both T -generic for all $a \in Y$, by construction. We then have that $\#Y \geq q^{-2Ri} \#X$, and we have seen that we arrive in this case with $i \leq 2R$. This proves the lemma. \square

Before proceeding to the next section, which contains the proof of the main results, we give an explicit bound on the difference between the canonical local height $\lambda_\phi(x)$ and the naive local height $\log^+ |x^{-1}|$. This will be used below to estimate the difference between the naive and canonical global heights.

Lemma 4.10. *Let ϕ/\mathbb{C}_v be a Drinfeld module of rank r . We have, for all $x \in \mathbb{C}_v$, and any non-constant $T \in A$,*

$$\begin{aligned} (4) \quad & -\frac{|T|_\infty^r}{(|T|_\infty^r - 1)^2} \log^+ |T| - \frac{|T|_\infty^r}{|T|_\infty^r - 1} j_{\phi_T, v} - \max\{0, -c(\phi)\} \\ & \leq \lambda_\phi(x) - \log^+ |x^{-1}| \\ & \leq j_{\phi_T, v} + \frac{1}{|T|_\infty^r - 1} \log^+ |T| + \max\{0, c(\phi)\}. \end{aligned}$$

Proof. Write

$$\phi_T(x) = Tx + a_1 x^q + \cdots + \Delta a^{q^{r \deg(T)}},$$

consider first the case in which $\Delta = 1$, and write $R = r \deg(T)$ as usual. By the triangle inequality, we have

$$\begin{aligned} \log |\phi_T(x)| &\leq \log \max \left\{ |Tx|, |a_1 x^q|, \dots, |x^{q^R}| \right\} \\ &\leq \log \max \{ |T|, |a_1|, \dots, 1 \} + q^R \log^+ |x| \\ &\leq \log \max \{ |a_1|, \dots, 1 \} + \log^+ |T| + q^R \log^+ |x| \\ &\leq (q^R - 1)j_{\phi_T, v} + \log^+ |T| + q^R \log^+ |x| \end{aligned}$$

It follows by induction that

$$G_\phi(x) - \log^+ |x| \leq j_{\phi_T, v} + \frac{1}{q^R - 1} \log^+ |T|.$$

On the other hand, we have

$$\log B_T = \max_{0 \leq i \leq R} \left\{ 0, \frac{1}{q^R - q^i} \log |a_i| \right\} \leq \frac{1}{q^R - 1} \log^+ |T| + j_{\phi_T, v},$$

since $q \geq 2$. If $\log |x| > \log B_T$, then we have

$$q^{-R} \log |\phi_T(x)| = q^{-R} \log |x^{q^R}| = \log |x|.$$

If, on the other hand, $\log |x| \leq \log B_T$, then we have

$$q^{-R} \log^+ |\phi(x)| - \log^+ |x| \geq -\log B_T \geq -\frac{1}{q^R - 1} \log^+ |T| - j_{\phi_T, v}.$$

By induction, we have

$$\begin{aligned} q^{-nR} \log^+ |\phi_T(x)| - \log^+ |x| &\geq -\left(1 + \frac{1}{q^R} + \dots\right) \left(\frac{1}{q^R - 1} \log^+ |T| + j_{\phi_T, v}\right) \\ &= -\frac{q^R}{(q^R - 1)^2} \log^+ |T| - \frac{q^R}{q^R - 1} j_{\phi_T, v}. \end{aligned}$$

So we have

$$-\frac{q^R}{(q^R - 1)^2} \log^+ |T| - \frac{q^R}{q^R - 1} j_{\phi_T, v} \leq G_\phi(x) - \log^+ |x| \leq j_{\phi_T, v} + \frac{1}{q^R - 1} \log^+ |T|.$$

Now, if $\Delta \neq 1$, we may apply the above bound to obtain a result for $\psi = \alpha \phi \alpha^{-1}$, where $\alpha^{q^R - 1} = \Delta$. Replacing x with αx in the above, and noting that $G_\phi(x) = G_\psi(\alpha x)$, $j_{\phi_T, v} = j_{\psi_T, v}$ and $c(\phi) = -\log |\alpha|$ we have

$$\begin{aligned} -\frac{q^R}{(q^R - 1)^2} \log^+ |T| - \frac{q^R}{q^R - 1} j_{\phi_T, v} - \log^+ |\alpha^{-1}| \\ \leq G_\phi(x) - \log^+ |x| \\ \leq j_{\phi_T, v} + \frac{1}{q^R - 1} \log^+ |T| + \log^+ |\alpha|. \end{aligned}$$

from the inequality

$$\log^+ |x| - \log^+ |\alpha^{-1}| \leq \log^+ |\alpha x| \leq \log^+ |x| + \log^+ |\alpha|.$$

We now note that $\log |\alpha| = -c(\phi)$, and that

$$\lambda_\phi(x) - \log^+ |x^{-1}| = G_\phi(x) - \log^+ |x| + c(\phi),$$

and hence we have (4). \square

In Lemma 4.3 we saw that $\lambda_\phi(x)$ is non-negative in the case of good reduction. To establish Theorem 1.1, however, we will need a stronger statement in the case of bad reduction which is potentially good, since in this case we have $\mathcal{D}_{\phi,v} > 0$.

Lemma 4.11. *Suppose that v is a finite place, that ϕ/L has potentially good reduction, and that $\mathcal{D}_{\phi,v} > 0$. Then we have*

$$\lambda_\phi(x) + G_\phi(x) \geq \frac{1}{d-1} \mathcal{D}_{\phi,v}$$

for all $x \in \phi(L)$, where $d(A, r)$ is as defined in Proposition 3.6.

Proof. Assume, without loss of generality, that ϕ is a minimal integral model of itself, and choose an $\alpha \in \mathbb{C}_v$ such that $\psi = \alpha\phi\alpha^{-1}$ has good reduction. Note that, in this case, $\mathcal{D}_{\phi/L} = c(\phi) = -\log|\alpha|$, since $c(\psi) = 0$. Furthermore, since ψ has good reduction, we see that $j_{\phi,v} = j_{\psi,v} = 0$, and hence by Proposition 3.6 it must hold that $0 < v(\alpha) < 1$ (the case $v(\alpha) = 0$ is excluded by our hypotheses).

From the proof of Lemma 4.3, we see that $\lambda_\phi(x) = \lambda_\psi(\alpha x)$ for all $x \in L$, and $G_\phi(x) = G_\psi(\alpha x)$. First suppose that $|\alpha x| \leq 1$. Then, since $v(x) \in \mathbb{Z}$ and $0 < v(\alpha) < 1$, we have

$$\lambda_\phi(x) = \lambda_\psi(\alpha x) = \log^+ |\alpha^{-1}x^{-1}| = (v(\alpha) + v(x)) \deg(v) \geq v(\alpha) \deg(v) = \mathcal{D}_{\phi,v}.$$

The claimed inequality follows from the fact that $G_\phi(x) \geq 0$ for all x .

On the other hand, suppose that $|\alpha x| > 1$, and note that $0 < v(\alpha) < 1$ is a rational number with denominator at most $d = d(A, r)$. It follows, since $v(x)$ is an integer for $x \in L$, that $v(\alpha x) < 0$ implies

$$v(\alpha x) \leq v(\alpha) - 1 \leq -\frac{1}{d} \leq -\frac{1}{d-1} v(\alpha).$$

Since ψ has good reduction, we obtain

$$G_\phi(x) = G_\psi(\alpha x) = \log^+ |\alpha x| \geq \frac{1}{d-1} \log |\alpha| = \frac{1}{d-1} \mathcal{D}_{\phi,v}.$$

□

Finally, we prove a lower bound on the Green's function associated to ϕ , to be used in Section 5.

Lemma 4.12. *If $|x| > B_T$, then*

$$G_\phi(x) \geq \left(\frac{q-1}{q^{r \deg(T)} - 1} \right) j_{\phi_T, v}.$$

Proof. It follows from Lemma 3.1 (iii) that for all x with $|x| > B_T$, we have

$$G_\phi(x) = \log |x| - c(\phi).$$

Note that, by changing coordinates, it suffices to prove the statement in the case where $\Delta = 1$. Then we have, for $|x| > \log B_T$,

$$\begin{aligned}
 G_\phi(x) &= \log |x| - c(\phi) \\
 &> \max \left\{ \frac{1}{q^{r \deg(T)} - 1} \log |T|, \frac{1}{q^{r \deg(T)} - q} \log |a_1|, \dots, \frac{1}{q^{r \deg(T)} - q^{r \deg(T)-1}} \log |a_{r \deg(T)-1}|, 0 \right\} \\
 &\geq \max \left\{ \frac{1}{q^{r \deg(T)} - q} \log |a_1|, \dots, \frac{1}{q^{r \deg(T)} - q^{r \deg(T)-1}} \log |a_{r \deg(T)-1}|, 0 \right\} \\
 &\geq \frac{q-1}{q^{r \deg(T)} - q} \max \left\{ \frac{1}{q-1} \log |a_1|, \dots, \frac{1}{q^{r \deg(T)-1} - 1} \log |a_{r \deg(T)-1}|, 0 \right\} \\
 &= \left(\frac{q-1}{q^{r \deg(T)} - 1} \right) j_{\phi_T, v}.
 \end{aligned}$$

□

5. DRINFELD MODULES OVER GLOBAL FIELDS

Throughout this section, L is a global A -field, as defined in Section 2, and M_L is its set of places. For any place $v \in M_L$, we will denote by L_v the completion of L . If ϕ/L is a Drinfeld module, then ϕ naturally gives rise to a Drinfeld module over L_v , which we will also call ϕ . The objects $c(\phi)$, B_T , G_ϕ , and λ_ϕ defined in Section 3 for ϕ/L_v will now be denoted by $c_v(\phi)$, $B_{T,v}$, $G_{\phi,v}$ and $\lambda_{\phi,v}$, to emphasize the dependence on the particular place $v \in M_L$, and similarly for finite extensions. We recall the canonical height defined by Denis [5], which decomposes (as per Poonen [13]) as a sum of Green's functions.

$$\hat{h}_\phi(x) = \sum_{v \in M_{L'}} \frac{[L'_v : L_v]}{[L' : L]} G_{\phi,v}(x).$$

Note that it follows immediately from the product formula that

$$(5) \quad \hat{h}_\phi(x) = \sum_{v \in M_L} \frac{[L'_v : L_v]}{[L' : L]} \lambda_{\phi,v}(x),$$

so long as $x \neq 0$. Again, this gives a well-defined function $\hat{h}_\phi : \phi(L^c) \rightarrow \mathbb{R}$.

In his original construction of the canonical height associated to a Drinfeld A -module, Denis [5] showed that this quantity differs from the Weil height by only a bounded amount. The purpose of the first result of this section is to make this bound explicit, in terms of various quantities related to ϕ . We define, for a finite extension L'/L and a Drinfeld module ϕ/L' ,

$$h(\phi) = h(j_\phi) + \sum_{v \in M_{L'}} \frac{[L'_v : L_v]}{[L' : L]} \max\{0, c_v(\phi)\}.$$

Note that if $T \in A$ is any non-constant element, and

$$\phi_T(x) = Tx + \dots + a_{r \deg(T)} x^{q^{r \deg(T)}},$$

then we have

$$\sum_{v \in M_{L'}} \frac{[L'_v : L_v]}{[L' : L]} \max\{0, c_v(\phi)\} = \frac{1}{|T|_\infty^{r \deg(T)} - 1} h(a_{r \deg(T)}),$$

and so $h(\phi)$ is just the sum of the heights of two quantities related to ϕ . To justify using the terminology of heights, we prove a Northcott-type result for this quantity.

Proposition 5.1. *For any quantities B and D , there are only finitely many Drinfeld modules ϕ/E such that E/L is a finite extension of degree at most D , and $h(\phi) \leq B$.*

Proof. First, note that if $h(\phi) \leq B$ and ϕ is defined over an extension of degree at most D of L , then j_ϕ is a point of bounded height and bounded algebraic degree, over L . In particular, there can be only finitely many \bar{L} -isomorphism classes of Drinfeld modules satisfying $h(\phi) \leq B$ defined over extensions of degree at most D . It suffices, then, to show that each such isomorphism class contains only finitely many instances ϕ with $h(\phi) \leq B$, and defined over extensions of degree at most D . But if ϕ is such an instance, and

$$\phi_{T_i}(x) = T_i x + a_{i,1}x^q + \cdots + a_{i,\deg(T_i)r}x^{q^{\deg(T_i)r}},$$

then $h(\phi) \leq B$ implies $h(a_{i,r\deg(T_i)}) \leq (q^{r\deg(T_i)} - 1)B$ for each i . In particular, the elements $a_{i,r\deg(T_i)}$ reside in a set of bounded height and degree, for each i . So there are only finitely many choices for each i . But suppose that $\phi\alpha = \alpha\psi$ for some $\alpha \in \bar{L}$, and that

$$\phi_{T_i}(x) = T_i x + b_{i,1}x^q + \cdots + a_{i,\deg(T_i)r}x^{q^{\deg(T_i)r}},$$

for all i . Then $\alpha^{q^{\deg(T_i)}-1} = 1$ for all i . In particular, there are only finitely many pairwise isomorphic ϕ/\bar{L} corresponding to a given choice of $j_{\phi,v}$ and a given choice of $a_{1,q\deg(T_1)}, \dots, a_{m,q\deg(T_m)}$. \square

We now state a slightly stronger version of Theorem 1.6, recalling that $L^{\text{Sep}} \subseteq L^c \subseteq \bar{L}$, with equality in the second inclusion when $L = K$, for example.

Theorem 5.2. *Let L'/L be a finite extension contained in L^c , and let ϕ/L' be a Drinfeld module of rank $r \geq 1$. There exist constants C_1 and C_2 , depending only on A , L , and r such that for all $x \in L^c$, we have*

$$-C_1 - \left(\frac{1}{1 - q^{-r}} \right) h(\phi) \leq \hat{h}_\phi(x) - h(x) \leq h(\phi) + C_2.$$

Proof. Let T_1, \dots, T_m generate A as an \mathbb{F}_q -algebra. By Lemma 4.10, we have the inequality (4) for each T_i , for each place $v \in L'$. Note that if v is a finite place, then $j_{\phi_{T_i},v} = j_{\phi,v}$ for each i , and $\log^+ |T_i|_v = 0$, and so we have

$$-\frac{1}{1 - |T_i|_\infty^{-r}} j_{\phi,v} - \max\{0, -c_v(\phi)\} \leq \lambda_\phi(x) - \log^+ |x^{-1}| \leq j_{\phi,v} + \max\{0, c_v(\phi)\}$$

for each such place. Note that, since $j_{\phi,v} \geq 0$, we may weaken the lower bound slightly by replacing $1/(1 - |T_i|_\infty^{-r})$ by $1/(1 - q^{-r})$, and similarly for the term $\max\{0, -c_v(\phi)\}$. Thus we have

$$-\frac{1}{1 - q^{-r}} (j_{\phi,v} + \max\{0, -c_v(\phi)\}) \leq \lambda_\phi(x) - \log^+ |x^{-1}| \leq j_{\phi,v} + \max\{0, c_v(\phi)\}$$

at every finite place v . Note that if L has no infinite places, we may sum this over $v \in M_L$ to obtain the claimed inequality with $C_1 = C_2 = 0$.

At every infinite place, we similarly have

$$\begin{aligned} & -\frac{|T_i|_\infty^r}{(|T_i|_\infty^r - 1)^2} \log^+ |T_i|_v - \frac{1}{1 - q^{-r}} (j_{\phi_{T_i}, v} + \max\{0, -c_v(\phi)\}) \\ & \leq \lambda_\phi(x) - \log^+ |x^{-1}| \\ & \leq j_{\phi_{T_i}, v} + \max\{0, c_v(\phi)\} + \frac{1}{|T_i|_\infty^r - 1} \log^+ |T_i|_v. \end{aligned}$$

Since $j_{\phi, v} = \max_{1 \leq i \leq m} j_{\phi_{T_i}, v}$, summing over places gives the claimed bound with

$$C_1 = \sum_{i=1}^m \frac{|T_i|_\infty^r}{(|T_i|_\infty^r - 1)^2} \sum_{v \in M_L^\infty} \log^+ |T_i|_v$$

and

$$C_2 = \sum_{i=1}^m \frac{1}{|T_i|_\infty^r - 1} \sum_{v \in M_L} \log^+ |T_i|_v$$

□

Remark 5.3. Note that the constants C_1 and C_2 depend on L only because of the generality in which we are working. If we consider only those L which are finite extensions of K , it makes sense to consider all heights relative to K (in other words, to take K as the ground field). In this case the constants above depend only on A and r .

If, on the other hand, L/K is transcendental, then every place is a finite place, and the constants C_1 and C_2 vanish.

Note that if X/L is a curve, and $\phi/L(X)$ a Drinfeld module, then both j_ϕ and $a_{r \deg(T)}$, in the notation above, can be viewed as morphisms from X to $M_{A, r}$ and \mathbb{P}^1 , respectively. It follows that $h(\phi_t) = O(h(t))$.

Corollary 5.4. *Let X/L be a curve, let h be a height on X with respect to an ample divisor, and let $\phi/L(X)$ be a Drinfeld A -module. Then we have*

$$\hat{h}_{\phi_t}(x) = h(x) + O(h(t)),$$

where the implied constant depends only on the generic fibre ϕ and the choice of height h .

We now define the quantity $\mathcal{D}_{\phi/L}$ alluded to in the introduction, which is similar to the minimal discriminant used by Taguchi [27].

Definition 5.5. Let ϕ/L be a Drinfeld module and, for each $v \in M_L^0$, let $\mathcal{D}_{\phi, v}$ be the (local) minimal discriminant defined above. Then the *minimal discriminant* of ϕ is the formal \mathbb{Q} -linear combination of elements of M_L^0 given by

$$\mathcal{D}_{\phi, v} = \sum_{v \in M_L^0} \mathcal{D}_{\phi, v}[v].$$

We will define the *discriminant* of ϕ/L to be the quantity

$$\Delta_{\phi/L} = \sum_{v \in M_L^0} c_v(\phi)[v].$$

We will say that ϕ is a *global minimal model* if $\Delta_{\phi,v} = \mathcal{D}_{\phi,v}$, and that ϕ is *quasi-minimal* if ϕ is a v -integral model for every $v \in M_L^0$, and $\deg(\Delta_{\phi/L}) = \sum_{v \in M_L^0} c_v(\phi)$ is minimal within the L -isomorphism class of ϕ .

Note that, if $B \subseteq L$ is the ring of elements integral at every infinite place, then $\mathcal{D}_{\phi/L}$ and $\Delta_{\phi/L}$ can be interpreted as divisors on $\text{Spec}(B)$ with coefficients in \mathbb{Q} , and we will use this terminology. It is clear from the definition that $\mathcal{D}_{\phi/L} \leq \Delta_{\phi/L}$ if ϕ is defined over B , so minimal models are always quasi-minimal, but the converse might not hold. Indeed, suppose that ϕ/L is a Drinfeld module, and for each $v \in M_L^0$ choose an $\alpha_v \in L_v$ so that $\alpha_v \phi \alpha_v^{-1}$ is a (local) minimal model. Then ϕ/L has a global minimal model if and only if there is a $\beta \in L$ such that $|\beta|_v = |\alpha_v|_v$ for each finite place v . In other words, if we define the *Weierstrass divisor* of ϕ/L to be the divisor

$$\mathfrak{a}_{\phi/L} = \sum_{v \in M_L^0} v(\alpha_v)[v]$$

on $\text{Spec}(B)$, then the class of $\mathfrak{a}_{\phi/L}$ is an L -isomorphism invariant of ϕ , and is trivial if and only if ϕ admits a global minimal model over L .

Although not every Drinfeld module over every field admits a minimal model, we see below that, at least for finite extensions of K , quasi-minimal models are close to being minimal.

Proposition 5.6. *For any finite extension L/K and any quasi-minimal ϕ/L , we have*

$$\deg(\mathcal{D}_{\phi/L}) \leq \deg(\Delta_{\phi/L}) \leq \deg(\mathcal{D}_{\phi/L}) + g(L) + [L : K] - 1,$$

where $g(L)$ is the genus of L .

Proof. Note that $\Delta_{\phi/L} = \mathcal{D}_{\phi/L} + \mathfrak{a}_{\phi/L}$, where $\mathfrak{a}_{\phi/L}$ is the Weierstrass divisor defined above. The first inequality is obvious, since $\mathfrak{a}_{\phi/L} \geq 0$, and for the second inequality we wish to bound $\deg(\mathfrak{a}_{\phi/L})$ given that ϕ is quasi-minimal. Note that if $\beta \in L$ and $\psi\beta = \beta\phi$, then we have $\mathfrak{a}_{\phi/L} = \mathfrak{a}_{\psi/L} + (\beta)_{\text{fin}}$, where

$$(\beta)_{\text{fin}} = \sum_{v \in M_L^0} v(\beta)[v],$$

and $\Delta_{\phi/L} = \Delta_{\psi/L} + (\beta)_{\text{fin}}$.

Let $v \in M_L^\infty$ be any infinite place, and suppose that $\beta \in L \setminus \{0\}$ satisfies $(\beta) \leq \mathfrak{a}_{\phi/L} - [v]$, where (β) is the usual divisor associated to β . Then $(\beta)_{\text{fin}} \leq \mathfrak{a}_{\phi/L}$, since the latter divisor is supported on M_L^0 , and so if $\psi = \beta\phi\beta^{-1}$, we see that

$$\Delta_{\psi/L} = \mathcal{D}_{\phi/L} + \mathfrak{a}_{\phi/L} - (\beta)_{\text{fin}} \geq \mathcal{D}_{\phi/L}.$$

In particular, ψ/L is an integral model, and $\deg(\Delta_{\phi/L}) = \deg(\Delta_{\psi/L}) + \deg(\beta)_{\text{fin}}$. By the quasi-minimality of ψ , we must have $\deg(\beta)_{\text{fin}} \leq 0$. But we also have $(\beta) - (\beta)_{\text{fin}} \leq -[v]$, and so

$$0 = \deg((\beta)_{\text{fin}}) + \deg((\beta) - (\beta)_{\text{fin}}) \leq -\deg(v) < 0,$$

which is impossible.

In other words, we have shown that there is no element $\beta \in L \setminus \{0\}$ such that $(\beta^{-1}) + \mathfrak{a}_{\phi/L} - [v] \geq 0$, and so the Riemann-Roch space $\mathcal{L}(\mathfrak{a}_{\phi/L} - [v])$ is trivial. It follows that

$$\deg(\mathfrak{a}_{\phi/L}) \leq g(L) - 1 + \deg(v) \leq g(L) + [L : K] - 1,$$

where $g(L)$ is the genus of L . □

We may now commence with the technical lemmas needed for the proof of Theorem 1.1. If $b \in A$ and $C \geq 0$, the C -truncated ideal generated by b will be the set

$$I(b, C) = b \cdot \{a \in A : |a|_\infty \leq C\}.$$

Lemma 5.7. *Fix $x \in \phi(L)$, $B \geq 1$, $b \in A$, and let $S \subseteq M_L$ be a finite set of places. Then there exists a $B' \geq 1$ and $b' \in A$ with $I(b', B') \subseteq I(b, B)$ such that for each $v \in S$ we have either*

- (i) *for all $a \in I(b', B')$, $|\phi_{T^2a}(x)|_v \leq B_{T,v}$ or*
- (ii) *for all $a \in I(b', B')$, $|\phi_a(x)|_v > B_{T,v}$ or $a = 0$.*

Furthermore we may take $B' \geq B^{2^{-\#S}} |T|_\infty^{-2}$

Proof. Let $b_0 = b$, let $B_0 = B$, and let C be a constant to be determined later, so that

$$I(b_0, B_0) = I(b, B) = b \cdot \{a \in A : |a|_\infty \leq B\}.$$

Now, order the places $S = \{v_1, v_2, \dots, v_s\}$, where $s = \#S$.

Let $i \geq 0$, and suppose that for all $a \in I(b_i, B_i)$ with $|a|_\infty \leq |b_i|_\infty C B_i^{1/2}$ we have $|\phi_{T^2a}(x)|_v \leq B_{T,v}$. Then we set $b_{i+1} = b_i$ and $B_{i+1} = C B_i^{1/2} \leq B_i$. Note that $I(b_{i+1}, B_{i+1}) \subseteq I(b_i, B_i)$, and for all $a \in I(b_{i+1}, B_{i+1})$ we have $|\phi_{T^2a}(x)|_v \leq B_{T,v}$.

If this is not the case, then there exists an $a \in I(b_i, B_i)$ with $|a|_\infty \leq |b_i|_\infty C B_i^{1/2}$ and $|\phi_{T^2a}(x)|_v > B_{T,v}$. Then for all $d \in T^2aA$ we have either $d = 0$, or $|\phi_d(x)|_v > B_{T,v}$. Let $a_{i+1} = T^2a$ and $B_{i+1} = C B_i^{1/2}$. Note that by construction, for all $d \in I(b_{i+1}, B_{i+1})$, either $d = 0$ or else $|\phi_d(x)|_v > B_{T,v}$. Also, note that $I(b_{i+1}, B_{i+1}) \subseteq I(b_i, B_i)$, since $b_i \mid b_{i+1}$ and if $d = b_{i+1}c$, with $|c|_\infty \leq B_{i+1}$, then $d = b_i \cdot \frac{T^2a}{b_i} \cdot c$ with

$$\left| b_i \frac{T^2a}{b_i} c \right|_\infty \leq |b_i|_\infty |T|_\infty^2 C B_i^{1/2} C B_i^{1/2} \leq |b_i|_\infty B_i,$$

as long as $C = |T|_\infty^{-1}$, and so $d \in I(b_i, B_i)$.

Note that in either case, $B_{i+1} = |T|_\infty^{-1} B_i^{1/2}$, and so we have

$$B' = B_s \geq B_0^{2^{-s}} |T|_\infty^{-(1+\frac{1}{2}+\dots+\frac{1}{2^{s-1}})} \geq B^{2^{-s}} |T|_\infty^{-2}.$$

□

We are now in a position to prove Theorem 1.1. We prove the following slightly more general result, and then prove Theorem 1.1 below as a corollary.

Theorem 5.8. *Let L be a global A -field, and let $s, r \geq 1$. Then there are constants $\varepsilon > 0$ and C , and an ideal $\mathfrak{a} \subseteq A$, such that for every Drinfeld module ϕ/L of rank r with genuinely bad reduction at at most s places, we have either $x \in \phi[\mathfrak{a}]$ or else*

$$\hat{h}_\phi(x) \geq \varepsilon \max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\} - C$$

for each $x \in \phi(L)$.

Proof. Let $x \in \phi(L)$, let S be a set containing all of the places of persistently bad reduction of ϕ/L , including all infinite places, and fix a set of (non-constant) generators T_1, \dots, T_m for A as an \mathbb{F}_q -algebra. Note that, by Lemma 4.3 above we have $\lambda_{\phi,v}(x) \geq 0$ for all $v \notin S$, and all non-zero $x \in \phi(L)$. By Lemma 3.7 of [12],

we also see that if $v \in M_L^0$ and ϕ has potentially good reduction at v , then $j_{\phi,v} = 0$. In other words, for any $\varepsilon > 0$ we have

$$\lambda_{\phi,v}(x) + G_{\phi,v}(x) \geq 0 = \varepsilon j_{\phi,v}$$

for any $v \notin S$, and for any $x \in \phi(L)$. On the other hand, by Lemma 4.11 we also have

$$\lambda_{\phi,v}(x) + G_{\phi,v}(x) \geq \frac{1}{d(A,r) - 1} \mathcal{D}_{\phi,v}$$

for all places of potentially good reduction, and so

$$\lambda_{\phi,v}(x) + G_{\phi,v}(x) \geq 0 = \varepsilon \max\{j_{\phi,v}, \mathcal{D}_{\phi,v}\}$$

so long as $\varepsilon \leq 1/(d(A,r) - 1)$. The challenge is to obtain a lower bound similar to this at places $v \in S$.

We will fix some large value C_m , to be specified later, which will depend only on m , r , and $s = \#S$. Applying Lemma 5.7 to T_1 and to the set $I(1, C_m)$, we find a subset $I(b_1, C_{m-1})$ such that for all $v \in S$ we have either $|\phi_{T_1^2 a}(x)|_v \leq B_{T_1,v}$ for all $a \in I(b_1, C_{m-1})$, or we have $|\phi_a(x)|_v > B_{T_1,v}$ for all non-zero $a \in I(b_1, C_{m-1})$. Also, note that $C_{m-1} \geq C_m^{2^{-s}} |T_1|_{\infty}^{-2}$.

Now, construct $I(b_2, C_{m-2}) \subseteq I(b_1, C_{m-1})$ such that

$$C_{m-2} \geq C_{m-1}^{2^{-s}} |T_2|_{\infty}^{-2} \geq C_m^{2^{-2s}} |T_1|_{\infty}^{-2^{1-s}} |T_2|_{\infty}^{-2}$$

and such that for each $v \in S$ we have either $|\phi_{T_2^2 a}(x)|_v \leq B_{T_2,v}$ for $a \in I(b_2, C_{m-2})$ or we have $|\phi_a(x)|_v > B_{T_2,v}$ for all non-zero $a \in I(b_2, C_{m-2})$. Proceeding inductively, we find a $b_m \in A$ and C_0 such that for each $1 \leq i \leq m$ and for each $v \in S$, we have either $|\phi_{T_i^2 a}(x)|_v \leq B_{T_i,v}$ for $a \in I(b_m, C_0)$ or we have $|\phi_a(x)|_v > B_{T_i,v}$ for all non-zero $a \in I(b_m, C_0)$. We will denote by $S_{1,i}$ the set of places at which the first phenomenon occurs, and $S_{2,i}$ the set at which the second does, so that $S = S_{1,i} \cup S_{2,i}$ for any $1 \leq i \leq m$. Note also that $C_0 \geq C_m^{2^{-ms}} C'$, for some constant C' depending only on A and $s = \#S$. Note that $Y_{1,0} = I(b_m, C_0)$ is an additive subgroup of A , and we can ensure that $Y_{1,0}$ contains at least $2 \cdot q^{-4s(\deg(T_1) + \deg(T_2) + \dots + \deg(T_m))}$ elements simply by choosing C_m larger than some value which depends only on A , r , and s .

Now, for each $1 \leq i \leq m$, enumerate the places in $S_{1,i} = \{v_1, v_2, \dots, v_{k_i}\}$, and set $Y_{i,0} = Y_{i-1,k_{i-1}}$ (where $Y_{1,0}$ is as defined above). We may apply Lemma 4.9, for each $1 \leq t \leq k_i$ to obtain an additive subgroup $Y_{i,t} \subseteq Y_{i,t-1}$ such that $\#Y_{i,t} \geq q^{-4r^2 \deg(T_i)} \#Y_{i,t-1}$, and such that for all $a \in Y_{i,t}$ we have $\phi_a(x) = 0$ or

$$-\log |\phi_a(x)|_{v_t} + c_{v_t}(\phi) \geq (1 - q^{-1}) j_{\phi_{T_i}, v_t} - \frac{1}{q(q^{r \deg(T_i)} - 1)^2} \log^+ |T_i^{-1}|.$$

To recap, we have an additive subgroup $Y_{i,k_i} \subseteq A$, with

$$\#Y_{i,k_i} \geq q^{-4k_i r^2 \deg(T_i)} \#Y_{i,0} \geq q^{-4s \deg(T_i)} \#Y_{i,0},$$

such that for all $v \in S_{1,i}$ and $a \in Y_{i,k_i}$, either $\phi_a(x) = 0$ or else

$$\begin{aligned} \lambda_{\phi,v}(\phi_a(x)) &\geq (1 - q^{-1}) j_{\phi_{T_i}, v} - \frac{1}{q(q^{r \deg(T_i)} - 1)^2} \log^+ |T_i^{-1}| \\ G_{\phi,v}(\phi_a(x)) &\geq 0. \end{aligned}$$

But it is true, by Lemma 4.12, that for all $v \in S_{2,i}$ and $a \in Y_{i,k_i}$, we have $a = 0$ or

$$\begin{aligned}\lambda_{\phi,v}(\phi_a(x)) &= 0 \\ G_{\phi,v}(\phi_a(x)) &\geq \left(\frac{q-1}{q^{r \deg(T_i)} - 1} \right) j_{\phi_{T_i},v}.\end{aligned}$$

In particular, for all $v \in S$ and all $a \in Y_{m,k_m}$, we get $\phi_a(x) = 0$ or

$$\lambda_{\phi,v}(\phi_a(x)) + G_{\phi,v}(\phi_a(x)) \geq \left(\frac{q-1}{q^{r \deg(T_i)} - 1} \right) j_{\phi_{T_i},v} - \frac{1}{q(q^{r \deg(T_i)} - 1)^2} \log^+ |T_i^{-1}|$$

for all $1 \leq i \leq m$, and so

$$(6) \quad \lambda_{\phi,v}(\phi_a(x)) + G_{\phi,v}(\phi_a(x)) \geq \left(\frac{q-1}{q^R - 1} \right) j_{\phi,v} - \delta_v,$$

where $R = r \max_{1 \leq i \leq m} \deg(T_i)$ and where

$$\delta_v = \max_{1 \leq i \leq m} \frac{1}{q(q^{r \deg(T_i)} - 1)^2} \log^+ |T_i^{-1}|.$$

Note that, by Proposition 3.6, we have

$$j_{\phi,v} > \frac{1}{d+1} \max\{j_{\phi,v}, \mathcal{D}_{\phi,v}\}$$

when $j_{\phi,v} > 0$, and so (6) is no weaker than

$$\lambda_{\phi,v}(\phi_a(x)) + G_{\phi,v}(\phi_a(x)) \geq \left(\frac{q-1}{q^R - 1} \right) \left(\frac{1}{d+1} \right) \max\{j_{\phi,v}, \mathcal{D}_{\phi,v}\} - \delta_v.$$

We may now choose C_m large enough, in a way which depends only on A and $\#S$, so that $\#Y_{m,k_m} \geq 2$. If we let $a \in Y_{m,k_m}$ be non-zero, then summing over all places gives

$$\begin{aligned}|a|_\infty^r \hat{h}_\phi(x) &= \hat{h}_\phi(\phi_a(x)) \\ &= \sum_{v \in M_L} \frac{1}{2} (\lambda_{\phi,v}(\phi_a(x)) + G_{\phi,v}(\phi_a(x))) \\ &\geq \sum_{v \in M_L} \frac{1}{2} \left(\left(\frac{q-1}{q^R - 1} \right) \left(\frac{1}{d+1} \right) \max\{j_{\phi,v}, \mathcal{D}_{\phi,v}\} - \delta_v \right) \\ &\geq \frac{1}{2} \left(\frac{q-1}{q^R - 1} \right) \left(\frac{1}{d+1} \right) \max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\} - C,\end{aligned}$$

unless $\phi_a(x) = 0$, where

$$C \leq \frac{1}{2} \sum_{v \in M_L} \frac{1}{q(q^r - 1)^2} \log^+ |T_i^{-1}| \leq \frac{[L : K]}{q(q^r - 1)^2} \sum_{1 \leq i \leq m} \deg(T_i).$$

Since $a \in Y_{m,k_m} \subseteq I(1, C_m)$, we have that $|a|_\infty \leq C_m$, and the latter quantity is chosen depending only on $\#S$ and the ring A , and so we have the claimed inequality. If, on the other hand, we have $\phi_a(x) = 0$, where $a \in Y_{m,k_m} \setminus \{0\}$, then $x \in \phi[aA] \subseteq \phi[\mathfrak{a}]$, where \mathfrak{a} is the least common multiple of all ideal of the form bA with $|b|_\infty \leq C_m$. Since C_m depends only on $\#S$ and the ring A , so does the ideal \mathfrak{a} . \square

Theorem 1.1 is a corollary of Theorem 5.8, once the following lemma is observed.

Lemma 5.9. *Every L -isomorphism class of Drinfeld modules contains an integral model ϕ/L such that*

$$(7) \quad h(\phi) \leq 2 \max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\} + g(L) + [L : K] - 1,$$

where $g(L)$ is the genus of L . In particular, for any B , there are at most finitely many L -isomorphism classes of Drinfeld modules ϕ/L of rank r satisfying

$$\max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\} \leq B.$$

Proof. The proof is similar to that of Proposition 5.6, but slightly different as the definition of quasi-minimality takes into account the value of $c_v(\phi)$ only at finite places, whereas the definition of $h(\phi)$ involves infinite places, as well.

Suppose that ϕ/L is an integral model such that $h(\phi)$ is minimal within the L -isomorphism class of ϕ , and let

$$D = \sum_{v \in M_L} \frac{\max\{0, c_v(\phi)\}}{\deg(v)} [v] - \mathcal{D}_{\phi/L} \geq 0.$$

Since ϕ is an integral model, $c_v(\phi) \geq 0$ for all $v \in M_L^0$. Notice, though, that if $c_v(\phi) \geq 0$ for all $v \in M_L^\infty$, then $c_v(\phi) = 0$ for all $v \in M_L$, by the product formula, hence $h(\phi) = h(j_\phi)$ and inequality (7) is trivial. There is no loss of generality, then, if we suppose that there is some $w \in M_L^\infty$ such that $c_w(\phi) < 0$.

Now suppose that there exists a $\beta \in L \setminus \{0\}$ satisfying $(\beta) \leq D - [w]$, and let $\psi = \beta\phi\beta^{-1}$. For every finite place v , we have

$$c_v(\psi) = c_v(\phi) - v(\beta) \deg(v) \geq \mathcal{D}_{\phi,v},$$

and so ψ is v -integral. We also have, for any $v \in M_L$,

$$\max\{0, c_v(\psi)\} + v(\beta) \deg(v) = \max\{v(\beta) \deg(v), c_v(\phi)\} \leq \max\{0, c_v(\phi)\}.$$

For the place w , we have the strict inequality

$$\max\{0, c_w(\psi)\} + w(\beta) \deg(w) = \max\{w(\beta) \deg(w), c_w(\phi)\} < 0 = \max\{0, c_w(\phi)\},$$

since both $c_w(\phi)$ and $w(\beta)$ are negative. By the product rule, we have

$$\begin{aligned} h(\psi) &= h(j_\psi) + \sum_{v \in M_L} \max\{0, c_v(\psi)\} \\ &= h(j_\phi) + \sum_{v \in M_L} (\max\{0, c_v(\psi)\} + v(\beta) \deg(v)) \\ &< h(j_\phi) + \sum_{v \in M_L} \max\{0, c_v(\phi)\} = h(\phi), \end{aligned}$$

a contradiction.

We have shown, then, that if ϕ/L is an integral model such that $h(\phi)$ is minimal with the L -isomorphism class, then the Riemann-Roch space $\mathcal{L}(D - [w])$ is trivial, and hence $\deg(D - [w]) \leq g(L) - 1$. From this we conclude that

$$\begin{aligned} h(\phi) &= h(j_\phi) + \sum_{v \in M_L} \max\{0, c_v(\phi)\} \\ &= h(j_\phi) + \deg(\mathcal{D}_{\phi/L}) + \deg(D) \\ &\leq 2 \max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\} + g(L) + \deg(w) - 1 \\ &\leq 2 \max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\} + g(L) + [L : K] - 1. \end{aligned}$$

For the second claim, that there are only finitely many L -isomorphism classes satisfying

$$\max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\} \leq B,$$

we may now simply invoke Proposition 5.1. \square

Proof of Theorem 1.1. Under the hypotheses of Theorem 1.1, we have by Theorem 5.8 constants ε and C and an ideal $\mathfrak{a} \subseteq A$ such that for every ϕ/L of rank r with at most s places of bad reduction, we have have either $x \in \phi[\mathfrak{a}]$ or else

$$\hat{h}_\phi(x) \geq \varepsilon \max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\} - C,$$

for each $x \in \phi(L)$. The latter case implies

$$\hat{h}_\phi(x) \geq \frac{\varepsilon}{2} \max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\},$$

unless ϕ/L satisfies $\max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\} \leq 2C/\varepsilon$. By Lemma 5.9, there are only finitely many L -isomorphism classes of ϕ/L satisfying this inequality.

Now, for each of these finitely many L -isomorphism classes, since the annihilator of the torsion submodule of $\phi(L)$ and the minimum positive value of \hat{h}_ϕ on $\phi(L)$ are both L -isomorphism invariants, we may adjust ε downward and \mathfrak{a} upward (in the sense of divisibility) so that it still true that

$$\hat{h}_\phi(x) \geq \frac{\varepsilon}{2} \max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\},$$

unless $x \in \phi[\mathfrak{a}]$. This proves Theorem 1.1. \square

We conclude this section with a proof of Theorem 1.4.

Proof of Theorem 1.4. Note that both the minimal positive value of $\hat{h}_\phi(x)$, as $x \in \phi(L)$ varies over non-torsion points, and the quantity $\max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\}$ are L -isomorphism invariants. In particular, there is no loss of generality if we prove Theorem 1.4 only in the case where ϕ/L is such that $h(\phi) \leq h(\psi)$ for any ψ/L which is L -isomorphic to ϕ .

Now, it follows from the assumption of uniform boundedness of torsion that there is a constant B , depending only on L and r , such that every Drinfeld module ϕ/L of rank r has a non-torsion point $x \in \phi(L)$ satisfying $h(x) \leq B$. Indeed, if we assume that $\#\phi^{\text{Tors}}(L) \leq N$ for every ϕ/L of rank r , then it suffices to choose B large enough that L contains more than N elements of height at most B . For quasi-minimal ϕ/L , it follows from this, Theorem 1.6, and Lemma 5.9 that there exists a non-torsion $x \in \phi(L)$ such that

$$\hat{h}_\phi(x) \leq h(x) + h(\phi) + O(1) \leq h(\phi) + O(1) \leq 2 \max\{h(j_\phi), \deg(\mathcal{D}_{\phi/L})\} + O(1),$$

where the implied constants depend only on A , r , and L . \square

6. DRINFELD MODULES IN FAMILIES

In this section we turn our attention to the proof of Theorem 1.5. The following fact is a modification of a result in [11], and the proof is nearly identical.

Theorem 6.1. *Let L be a global A -field, let X/L^c be a curve, let $\phi/L^c(X)$ be a Drinfeld A -module, and let $x \in \phi(L^c(X))$. Then there exists a divisor $D \in \text{Pic}(X) \otimes \mathbb{Q}$ such that for all $\beta \in X(L^c)$,*

$$\hat{h}_{\phi_\beta}(x_\beta) = h_D(\beta) + O(1).$$

Furthermore, $\deg(D) = \hat{h}_\phi(x) = 0$ only if $x \in \phi^{\text{Tors}}(L^c)$, or ϕ is isotrivial.

As noted, the proof of this result is essentially contained in [11]. Note that the arguments in [11] assume the context of a number field, but it is only in proving the local analyticity of local heights that the characteristic of the field enters into the argument. The fact that $\deg(D) = \hat{h}_\phi(x)$ follows directly from the construction in [11], and the fact that this vanishes only if $x \in \phi^{\text{Tors}}(L^c)$, or ϕ is isotrivial follows from a result of Ghioca [7] (see also Baker [1]).

Proof of Theorem 1.5. By the main result of [11], there is an effective divisor D on C of degree $\hat{h}_\phi(x)$ such that

$$\hat{h}_{\phi_\beta}(x_\beta) = h_D(\beta) + O(1),$$

and D is ample as long as ϕ is not isotrivial. The set of points at which the left-hand side vanishes, then, is a set of bounded height (with respect to the ample divisor D).

On the other hand, let E be the pole divisor of j_ϕ , so that ϕ_β has bad reduction only at places such that β is not integral with respect to E . If we fix this number of places, we have for any degree one height h on C ,

$$\varepsilon \deg(j_\phi)h(t) = \varepsilon h(j_{\phi_t}) \leq \hat{h}_{\phi_t}(y) + \kappa,$$

for all $y \in L$, unless $y \in \phi_\beta^{\text{Tors}}(L)$. So if $y \in \Gamma_{\phi_\beta}(x_\beta, L)$, then either $y \in \phi_\beta^{\text{Tors}}(L)$ or $\phi_{\beta,b}(y) = x_\beta$, for some $b \in A$. Then we have

$$\varepsilon \deg(j_\phi)h(t) \leq \hat{h}_{\phi_\beta}(y) + \kappa = |b|_\infty^{-r} \hat{h}_{\phi_\beta}(x_\beta) + O(1) = |b|_\infty^{-r} h_D(\beta) + O(1).$$

Dividing by $h(t)$, we have

$$|b|_\infty^r \leq \frac{\deg(D)}{\varepsilon \deg(j_\phi)} + o(1),$$

where $o(1) \rightarrow 0$ as $h(t) \rightarrow \infty$. This gives us only finitely many choices for b , once we exclude parameters coming from some set of bounded height.

So we've shown that if $y \in \Gamma_{\phi_\beta}(x_\beta, L)$, then either $\phi_{\beta,b}(y) \in \langle x_\beta \rangle$ for some b of degree bounded in terms of the number of places at which β is not integral with respect to E , or else $y \in \phi_\beta^{\text{Tors}}(L)$. But the latter is bounded in size by the number of places at which β is not integral with respect to E , and so the index $(\Gamma_{\phi_\beta}(x_\beta, L) : \langle x_\beta \rangle)$ is similarly bounded. \square

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